

ALGEBRAS OF SYMBOLS ASSOCIATED WITH THE WEYL CALCULUS FOR LIE GROUP REPRESENTATIONS

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ABSTRACT. We develop our earlier approach to the Weyl calculus for representations of infinite-dimensional Lie groups by establishing continuity properties of the Moyal product for symbols belonging to various modulation spaces. For instance, we prove that the modulation space of symbols $M^{\infty,1}$ is an associative Banach algebra and the corresponding operators are bounded. We then apply the abstract results to two classes of representations, namely the unitary irreducible representations of nilpotent Lie groups, and the natural representations of the semidirect product groups that govern the magnetic Weyl calculus. The classical Weyl-Hörmander calculus is obtained for the Schrödinger representations of the finite-dimensional Heisenberg groups, and in this case we recover the results obtained by J. Sjöstrand in 1994.

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1. INTRODUCTION

A quite important class of symbols for pseudo-differential operators was introduced by J. Sjöstrand in [Sj94] (see also [Sj95]). He denoted this class by $S(1)$ and pointed out that it has a number of remarkable properties, such as:

- (1) For every symbol $a \in S(1)$ the corresponding operator $\text{Op}(a)$ obtained by the pseudo-differential Weyl calculus is bounded on $L^2(\mathbb{R}^n)$.
- (2) The class $S(1)$ has a natural structure of unital involutive associative Banach algebra such that the mapping $\text{Op}: S(1) \rightarrow \mathcal{B}(L^2(\mathbb{R}^n))$ is a continuous $*$ -homomorphism.
- (3) If a symbol $a \in S(1)$ has the property that the operator $\text{Op}(a)$ is invertible in $\mathcal{B}(L^2(\mathbb{R}^n))$, then there exists $b \in S(1)$ such that $\text{Op}(a)^{-1} = \text{Op}(b)$.

Date: February 7, 2011.

2000 Mathematics Subject Classification. Primary 47G30; Secondary 22E25, 22E65, 47G10.

Key words and phrases. Weyl calculus; involutive Banach algebra; Wiener property; Lie group; modulation spaces.

It was later realized that the class $S(1)$ is actually the modulation space $M^{\infty,1}(\mathbb{R}^{2n})$ (see for instance [Gr01] for a broad discussion), and thus the above three properties become as many statements in representation theory of the Heisenberg groups.

The aim of the present paper is to present the deep representation theoretic background of properties (1)–(3), in the sense that we obtain below, in Theorem 2.15, their appropriate versions for some representations of infinite-dimensional Lie groups and their localized Weyl calculus proposed in our earlier papers [BB09a] and [BB10c]. We then apply these abstract results to two classes of representations:

— Representations of some infinite-dimensional Lie groups constructed as semidirect products of Lie groups and invariant function spaces thereon. We have pointed out in [BB09a] and [BB10a] that these semidirect products are the symmetry groups of the magnetic Weyl calculus developed for instance in [MP04], [IMP07], [MP10] and [IMP10], and we thus find versions of the aforementioned properties in this setting.

— Unitary irreducible representations of arbitrary nilpotent Lie groups. The Weyl correspondence for these representations was developed by [Pe94], and we have later introduced in [BB10b] the modulation spaces in this framework and established continuity properties of the operators constructed by the corresponding Weyl calculus. In particular we found a space of symbols, which for the Heisenberg group reduces to Sjöstrand’s class, and gives rise to bounded operators. We now show that that space of symbols has all the above properties (1)–(3) in the case of an arbitrary irreducible representation of a nilpotent Lie group.

The present paper is a sequel to [BB10c] and relies on the methods developed there. In addition, we use some ideas contained in the deep analysis of Sjöstrand’s class in [Gr06].

2. SJÖSTRAND’S ALGEBRA OF SYMBOLS IN AN ABSTRACT SETTING

In this section we rely on some terminology and results of [BB10c]. For the reader’s convenience we recall in the first subsection (‘Preliminaries’) the framework and the basic ideas needed in the sequel. Sections 3 and 4 are devoted to discussing wide classes of examples satisfying the assumptions of this abstract framework.

Preliminaries.

Setting 2.1. Throughout this section we keep the following notation:

- (1) M is a locally convex Lie group (see [Ne06]) with a smooth exponential mapping $\exp_M: \mathbf{L}(M) = \mathfrak{m} \rightarrow M$.
- (2) $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ is a twice nuclearly smooth unitary representation ([BB10c, Def. 2.2]). For instance, π may be any unitary irreducible representation of a nilpotent Lie group. See Section 3 for another class of examples.
- (3) Ξ is a finite-dimensional vector space with a Lebesgue measure and Ξ^* is a manifold with a polynomial structure (see [Pe89]) and a Radon measure, endowed with a function $\langle \cdot, \cdot \rangle: \Xi^* \times \Xi \rightarrow \mathbb{R}$ which is linear in the second variable and such that the “Fourier transform”

$$\widehat{\cdot}: L^1(\Xi) \rightarrow L^\infty(\Xi^*), \quad b(\cdot) \mapsto \widehat{b}(\cdot) = \int_{\Xi} e^{-i\langle \cdot, x \rangle} b(x) dx$$

gives a linear topological isomorphism $\mathcal{S}(\Xi) \rightarrow \mathcal{S}(\Xi^*)$ and a unitary operator $L^2(\Xi) \rightarrow L^2(\Xi^*)$. The inverse of this transform is denoted by $a \mapsto \check{a}$.

- (4) $\theta: \Xi \rightarrow \mathfrak{m}$ is a linear mapping such that
- (a) π satisfies the orthogonality relations along θ ([BB10c, Def. 3.2]);
 - (b) π satisfies the density condition along θ ([BB10c, Def. 3.6]);
 - (c) the localized Weyl calculus for π along θ is regular ([BB10c, Def. 3.10]);
 - (d) for every $u \in U(\mathfrak{m}_{\mathbb{C}})$ the function $\|d\pi(u)\pi(\exp_M(\theta(\cdot)))\phi_0\|$ has polynomial growth. \square

Remark 2.2. The setting of [BB10c] is actually slightly narrower in the sense that Ξ^* was supposed to be a finite-dimensional linear space and $\langle \cdot, \cdot \rangle: \Xi^* \times \Xi \rightarrow \mathbb{R}$ was supposed to be a duality pairing. However, it is easily seen that the above setting ensures that the main results of [BB10c] hold true. \square

Notation 2.3. Here we summarize some additional notation related to the above setting:

- (1) \mathcal{H}_{∞} is the space of smooth vectors for the representation π , which has a structure of nuclear Fréchet space.
- (2) $\mathcal{H}_{-\infty}$ is the space of continuous antilinear functionals on \mathcal{H}_{∞} , endowed with the topology of uniform convergence on the bounded sets. There exist the dense embeddings $\mathcal{H}_{\infty} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}_{-\infty}$, and the duality pairing $(\cdot | \cdot): \mathcal{H}_{-\infty} \times \mathcal{H}_{\infty} \rightarrow \mathbb{C}$ agrees with the scalar product of \mathcal{H} .
- (3) $\mathcal{B}(\mathcal{H})_{\infty}$ is the space of smooth vectors for the unitary representation

$$\pi \otimes \bar{\pi}: M \times M \rightarrow \mathcal{B}(\mathfrak{S}_2(\mathcal{H})), \quad (\pi \otimes \bar{\pi})(m_1, m_2)T = \pi(m_1)T\pi(m_2)^{-1}.$$

This is also a nuclear Fréchet space since $\mathcal{B}(\mathcal{H})_{\infty} \simeq \mathcal{H}_{\infty} \widehat{\otimes} \mathcal{H}_{\infty}$ (see [BB10c, Eq. (2.1)]).

- (4) $\mathcal{A}_{\phi}^{\pi, \theta} f \in \mathcal{C}(\Xi) \cap \mathcal{S}'(\Xi)$ is the *ambiguity function* for the representation π along the mapping θ , defined for $\phi \in \mathcal{H}_{\infty}$ and $f \in \mathcal{H}_{-\infty}$ by the formula

$$\mathcal{A}_{\phi}^{\pi, \theta} f: \Xi \rightarrow \mathbb{C}, \quad (\mathcal{A}_{\phi}^{\pi, \theta} f)(\cdot) = (f | \pi(\exp_M(\theta(\cdot)))\phi).$$

- (5) $\mathcal{W}(f, \phi) \in \mathcal{S}'(\Xi^*)$ is the *cross-Wigner distribution* for π along θ , defined by the formula $\mathcal{W}(f, \phi) = \mathcal{A}_{\phi}^{\pi, \theta} f$ for $\phi \in \mathcal{H}_{\infty}$ and $f \in \mathcal{H}_{-\infty}$.
- (6) $M_{\phi}^{p, q}(\pi, \theta)$ are the *modulation spaces* constructed for $p, q \in [1, \infty]$ with respect to a decomposition into a direct sum of subspaces $\Xi = \Xi_1 \dot{+} \Xi_2$ and the *window vector* $\phi \in \mathcal{H}_{\infty} \setminus \{0\}$. Specifically, for any measurable function $F: \Xi \simeq \Xi_1 \times \Xi_2 \rightarrow \mathbb{C}$ define

$$\|F\|_{L^{p, q}(\Xi_1 \times \Xi_2)} = \left(\int_{\Xi_1} \left(\int_{\Xi_2} |F(X_1, X_2)|^p dX_2 \right)^{q/p} dX_1 \right)^{1/q} \in [0, \infty]$$

with the usual conventions if p or q is infinite. Then

$$M_{\phi}^{p, q}(\pi, \theta) := \{f \in \mathcal{H}_{-\infty} \mid \|f\|_{M_{\phi}^{p, q}(\pi, \theta)} := \|\mathcal{A}_{\phi}^{\pi, \theta} f\|_{L^{p, q}(\Xi_1 \times \Xi_2)} < \infty\}.$$

If $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, then

$$M_{\phi}^{p_1, q_1}(\pi, \theta) \cap M_{\phi}^{\infty, \infty}(\pi, \theta) \subseteq M_{\phi}^{p_2, q_2}(\pi, \theta) \cap M_{\phi}^{\infty, \infty}(\pi, \theta), \quad (2.1)$$

since $L^{p_1, q_1}(\Xi_1 \times \Xi_2) \cap L^{\infty}(\Xi) \subseteq L^{p_2, q_2}(\Xi_1 \times \Xi_2) \cap L^{\infty}(\Xi)$. \square

Remark 2.4. We recall that the *localized Weyl calculus* for π along θ is the mapping $\text{Op}^\theta: \widehat{L^1(\Xi)} \rightarrow \mathcal{B}(\mathcal{H})$ given by

$$\text{Op}^\theta(a) = \int_{\Xi} \check{a}(X) \pi(\exp_M(\theta(X))) dX \quad (2.2)$$

for $a \in \widehat{L^1(\Xi)}$, where we use weakly convergent integrals. Under the present assumptions, it follows by [BB10c, Prop. 3.12] that the localized Weyl calculus defines a linear topological isomorphism $\text{Op}^\theta: \mathcal{S}(\Xi^*) \rightarrow \mathcal{B}(\mathcal{H})_\infty \simeq \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_\infty)$, its dual topological isomorphism $\text{Op}^\theta: \mathcal{S}'(\Xi^*) \rightarrow \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_\infty)$, and the mapping

$$\text{Op}^\theta: L^2(\Xi^*) \rightarrow \mathfrak{S}_2(\mathcal{H}), \quad (2.3)$$

which is a unitary operator.

If $a, b \in \mathcal{S}'(\Xi^*)$ and the operator product $\text{Op}^\theta(a)\text{Op}^\theta(b) \in \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_\infty)$ is well defined, then the *Moyal product* $a \#^\theta b \in \mathcal{S}'(\Xi^*)$ is uniquely determined by the condition

$$\text{Op}^\theta(a \#^\theta b) = \text{Op}^\theta(a)\text{Op}^\theta(b).$$

Thus the Moyal product defines bilinear mappings $\mathcal{S}(\Xi^*) \times \mathcal{S}(\Xi^*) \rightarrow \mathcal{S}(\Xi^*)$ and $L^2(\Xi^*) \times L^2(\Xi^*) \rightarrow L^2(\Xi^*)$. \square

Definition 2.5. Let us consider the semi-direct product $M \ltimes M$ defined by the action of M on itself by inner automorphisms. Thus $M \ltimes M$ is a locally convex Lie group whose underlying manifold is $M \times M$ and the group operation is

$$(m_1, m_2)(n_1, n_2) = (m_1 n_1, n_1^{-1} m_2 n_1 n_2)$$

for all $m_1, m_2, n_1, n_2 \in M$. There exists the natural continuous unitary representation

$$\pi^\ltimes: M \ltimes M \rightarrow \mathcal{B}(\mathfrak{S}_2(\mathcal{H})), \quad \pi^\ltimes(m_1, m_2)T = \pi(m_1 m_2)T\pi(m_1)^{-1}.$$

By using the unitary operator (2.3), we can construct the unitarily equivalent representation $\pi^\# : M \ltimes M \rightarrow \mathcal{B}(L^2(\Xi^*))$ (see [BB10c, Def. 3.13]). \square

Remark 2.6. The representation $\pi^\# : M \ltimes M \rightarrow \mathcal{B}(L^2(\Xi^*))$ has the following useful properties.

- (1) For every $X_1, X_2 \in \Xi$ and $f \in L^2(\Xi^*)$ we have [BB10c, Remark 3.14(2)]

$$\pi^\#(\exp_{M \ltimes M}(\theta(X_1), \theta(X_2)))f = e^{i\langle \cdot, X_1 + X_2 \rangle} \#^\theta f \#^\theta e^{-i\langle \cdot, X_1 \rangle}. \quad (2.4)$$

- (2) For $F \in \mathcal{S}'(\Xi^*)$ and $\Phi \in \mathcal{S}(\Xi^*)$ we can consider the ambiguity function $\mathcal{A}_\Phi^{\pi^\#, \theta \times \theta}: \Xi \times \Xi \rightarrow \mathbb{C}$ of the representation $\pi^\#$ along the linear mapping $\theta \times \theta: \Xi \times \Xi \rightarrow \mathfrak{m} \ltimes \mathfrak{m}$, just as in Notation 2.3(4), by the formula

$$(\mathcal{A}_\Phi^{\pi^\#, \theta \times \theta} F)(X_1, X_2) = (F | \pi^\#(\exp_{M \ltimes M}(\theta(X_1), \theta(X_2))\Phi))$$

for all $X_1, X_2 \in \Xi$. We also define

$$\|F\|_{M_\Phi^{r,s}(\pi^\#, \theta \times \theta)} = \left(\int_{\Xi} \left(\int_{\Xi} |(\mathcal{A}_\Phi^{\pi^\#, \theta \times \theta} F)(X_1, X_2)|^r dX_1 \right)^{s/r} dX_2 \right)^{1/s} \in [0, \infty]$$

with the usual conventions if r or s is infinite. The space

$$M_\Phi^{r,s}(\pi^\#, \theta \times \theta) := \{F \in \mathcal{S}'(\Xi^*) \mid \|F\|_{M_\Phi^{r,s}(\pi^\#, \theta \times \theta)} < \infty\}$$

is a *modulation space* of symbols for the localized Weyl calculus Op^θ associated with the unitary representation $\pi: M \rightarrow \mathcal{B}(\mathcal{H})$ along with the linear mapping $\theta: \Xi \rightarrow \mathfrak{m}$ for the *window vector* $\Phi \in \mathcal{S}(\Xi^*) \setminus \{0\}$.

□

Ambiguity functions and matrix coefficients. Here is a general version of the usual covariance property of the cross-Wigner distribution (see [Gr01, Prop. 4.3.2]).

Lemma 2.7. *For all $f_1, f_2 \in \mathcal{H}_{-\infty}$ and $X_1, X_2 \in \Xi$ we have the equation*

$$\begin{aligned} \mathcal{W}(\pi(\exp_M(\theta(X_1)))f_1, \pi(\exp_M(\theta(X_2)))f_2) \\ = \pi^\#(\exp_{M \ltimes M}(\theta(X_2), \theta(X_1 - X_2)))\mathcal{W}(f_1, f_2) \end{aligned}$$

in $\mathcal{S}'(\Xi^*)$.

Proof. We have

$$\begin{aligned} \text{Op}^\theta(\mathcal{W}(\pi(\exp_M(\theta(X_1)))f_1, \pi(\exp_M(\theta(X_2)))f_2)) \\ = (\cdot \mid \pi(\exp_M(\theta(X_2)))f_2)\pi(\exp_M(\theta(X_1)))f_1 \\ = \pi(\exp_M(\theta(X_1)))((\cdot \mid f_2)f_1)\pi(\exp_M(\theta(-X_2))) \\ = \text{Op}^\theta(e^{i\langle \cdot, X_1 \rangle})\text{Op}^\theta(\mathcal{W}(f_1, f_2))\text{Op}^\theta(e^{-i\langle \cdot, X_2 \rangle}) \\ = \text{Op}^\theta(e^{i\langle \cdot, X_1 \rangle})\#^\theta \mathcal{W}(f_1, f_2)\#^\theta e^{-i\langle \cdot, X_2 \rangle} \\ = \text{Op}^\theta(\pi^\#(\exp_{M \ltimes M}(\theta(X_2), \theta(X_1 - X_2)))\mathcal{W}(f_1, f_2)), \end{aligned}$$

where the latter equality relies on (2.4).

Then the assertion follows since $\text{Op}^\theta: \mathcal{S}'(\Xi^*) \rightarrow \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$ is a linear topological isomorphism. □

Theorem 2.8. *If $\phi \in \mathcal{H}_\infty$ and $a \in \mathcal{S}'(\Xi^*)$, then for all $X_1, X_2 \in \Xi$ we have*

$$(\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^\#, \theta \times \theta} a)(X_1, X_2) = (\text{Op}^\theta(a)\phi_{X_1} \mid \phi_{X_1+X_2})$$

where we denote $\phi_X := \pi(\exp_M(\theta(X)))\phi \in \mathcal{H}_\infty$ for each $X \in \Xi$.

Proof. First note that

$$\begin{aligned} \pi(\exp_M(\theta(-X_1 - X_2)))\text{Op}^\theta(a)\pi(\exp_M(\theta(X_1))) \\ = \text{Op}^\theta(e^{-i\langle \cdot, X_1+X_2 \rangle})\#^\theta a\#^\theta e^{i\langle \cdot, X_1 \rangle} \\ = \text{Op}^\theta(\pi^\#(\exp_{M \ltimes M}(\theta(-X_1), \theta(-X_2)))a) \end{aligned}$$

by (2.4). Therefore

$$\begin{aligned} (\text{Op}^\theta(a)\phi_{X_1} \mid \phi_{X_1+X_2}) &= (\pi(\exp_M(\theta(-X_1 - X_2)))\text{Op}^\theta(a)\pi(\exp_M(\theta(X_1)))\phi \mid \phi) \\ &= (\text{Op}^\theta(\pi^\#(\exp_{M \ltimes M}(\theta(-X_1), \theta(-X_2)))a)\phi \mid \phi) \\ &= (\text{Op}^\theta(\pi^\#(\exp_{M \ltimes M}(\theta(-X_1), \theta(-X_2)))a) \mid (\cdot \mid \phi)\phi) \\ &= (\text{Op}^\theta(\pi^\#(\exp_{M \ltimes M}(\theta(-X_1), \theta(-X_2)))a) \mid \text{Op}^\theta(\mathcal{W}(\phi, \phi))) \\ &= (\pi^\#(\exp_{M \ltimes M}(\theta(-X_1), \theta(-X_2)))a \mid \mathcal{W}(\phi, \phi)) \\ &= (a \mid \pi^\#(\exp_{M \ltimes M}(\theta(X_1), \theta(X_2)))\mathcal{W}(\phi, \phi)) \\ &= (\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^\#, \theta \times \theta} a)(X_1, X_2), \end{aligned}$$

and this completes the proof. □

Corollary 2.9. *Let $\phi \in \mathcal{H}_\infty$ and denote $\phi_X := \pi(\exp_M(\theta(X)))\phi \in \mathcal{H}_\infty$ for each $X \in \Xi$. Also let $p, q \in [1, \infty]$. For $a \in \mathcal{S}'(\Xi^*)$ denote by B_a the set of all measurable functions $\beta: \Xi \rightarrow [0, \infty]$ satisfying the condition*

$$(\forall X \in \Xi) \quad \|(\text{Op}^\theta(a)\phi_\bullet \mid \phi_{\bullet+X})\|_{L^p(\Xi)} \leq \beta(X). \quad (2.5)$$

Also define

$$\beta_a: \Xi \rightarrow [0, \infty], \quad \beta_a(X) = \|(\text{Op}^\theta(a)\phi_\bullet \mid \phi_{\bullet+X})\|_{L^p(\Xi)}.$$

Then we have $a \in M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)$ if and only if $B_a \cap L^q(\Xi) \neq \emptyset$, and in this case $\beta_a \in B_a \cap L^q(\Xi)$ and

$$\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)} = \inf_{\beta \in B_a \cap L^q(\Xi)} \|\beta\|_{L^q(\Xi)} = \|\beta_a\|_{L^q(\Xi)}. \quad (2.6)$$

Proof. If $a \in M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)$, then the function $a_0: \Xi \rightarrow [0, \infty]$ defined by $a_0(Y) := \|(\mathcal{A}_{\mathcal{W}(\phi, \phi)}^{\pi^\#, \theta \times \theta} a)(\cdot, Y)\|_{L^p(\Xi)}$ has the property $a_0 \in L^q(\Xi)$ and moreover it follows at once by Theorem 2.8 that $\|(\text{Op}^\theta(a)\phi_\bullet \mid \phi_{\bullet+X_2})\|_{L^p(\Xi)} \leq a_0(X_2)$ for $X_2 \in \Xi$. Hence condition (2.7) is satisfied for $\beta := a_0$.

Conversely, if (2.5) holds, then we get by Theorem 2.8 again that for all $Y \in \Xi$ we have $a_0(Y) \leq \beta(Y)$, whence $\|a_0\|_{L^q(\Xi)} \leq \|\beta\|_{L^q(\Xi)} < \infty$, and then $a \in M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)$ and $\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)} \leq \|\beta\|_{L^q(\Xi)}$.

Equality (2.6) is a by-product of the above reasoning, hence the proof is complete. \square

Remark 2.10. By using Corollary 2.9 for $p = \infty$ we get the following abstract version of the *almost diagonalization theorem* established in [Gr06, Th. 3.2]:

Let $\phi \in \mathcal{H}_\infty$ and denote as above $\phi_X := \pi(\exp_M(\theta(X)))\phi \in \mathcal{H}_\infty$ for each $X \in \Xi$. For $a \in \mathcal{S}'(\Xi^*)$ let B_a be the set of all measurable functions $\beta: \Xi \rightarrow [0, \infty]$ satisfying the condition

$$(\forall X_1, X_2 \in \Xi) \quad |(\text{Op}^\theta(a)\phi_{X_1} \mid \phi_{X_2})| \leq \beta(X_1 - X_2). \quad (2.7)$$

Also define

$$\beta_a: \Xi \rightarrow [0, \infty], \quad \beta_a(X) = \sup_{Y \in \Xi} |(\text{Op}^\theta(a)\phi_{X+Y} \mid \phi_X)|.$$

Then we have $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, q}(\pi^\#, \theta \times \theta)$ if and only if $B_a \cap L^q(\Xi) \neq \emptyset$, and in this case $\beta_a \in B_a \cap L^q(\Xi)$ and

$$\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, q}(\pi^\#, \theta \times \theta)} = \inf_{\beta \in B_a \cap L^q(\Xi)} \|\beta\|_{L^q(\Xi)} = \|\beta_a\|_{L^q(\Xi)} \quad (2.8)$$

whenever $1 \leq q \leq \infty$. \square

Definition 2.11. Let $\phi \in \mathcal{H}_\infty$ with $\|\phi\| = 1$ and $\phi_X = \pi(\exp_M(\theta(X)))\phi \in \mathcal{H}_\infty$ for each $X \in \Xi$. For every $a \in \mathcal{S}'(\Xi^*)$ we define

$$C_a: \Xi \times \Xi \rightarrow \mathbb{C}, \quad C_a(X, Y) := (\text{Op}^\theta(a)\phi_X \mid \phi_Y)$$

and the integral operator in $L^2(\Xi)$ defined by the integral kernel C_a will be denoted by

$$T_a: \mathcal{D}(T_a) \rightarrow L^2(\Xi).$$

Let us also denote by $V := \mathcal{A}_\phi: \mathcal{H} \rightarrow L^2(\Xi)$ the isometry defined by the ambiguity functions. Note that the constant function $1 \in \mathcal{S}'(\Xi^*)$ gives rise to the orthogonal

projection $T_1 = T_1^* = (T_1)^2 \in \mathcal{B}(L^2(\Xi))$ with $\text{Ran } T_1 = \text{Ran } V$ and $T_1 T_a = T_a T_1 = T_a$ for every $a \in \mathcal{S}'(\Xi)$. \square

We now present the main idea which allows to use integral operators on Ξ for the study of operators $\text{Op}^\theta(a): \mathcal{H}_\infty \rightarrow \mathcal{H}_{-\infty}$, $a \in \mathcal{S}'(\Xi^*)$.

Lemma 2.12. *For arbitrary $a \in \mathcal{S}'(\Xi)$ we have*

$$\text{Op}^\theta(a) = V^* T_a V: \mathcal{D}(\text{Op}^\theta(a)) \rightarrow \mathcal{H}$$

on the domain

$$\mathcal{D}(\text{Op}^\theta(a)) = \{f \in \mathcal{H} \mid Vf \in \mathcal{D}(T_a)\}.$$

In particular, $T_a \in \mathcal{B}(L^2(\Xi))$ if and only if $\text{Op}^\theta(a) \in \mathcal{B}(\mathcal{H})$.

Proof. Use Definition 2.11 along with the fact that the representation $\pi \otimes \bar{\pi}: M \times M \rightarrow \mathcal{B}(\mathfrak{S}_2(\mathcal{H}))$ satisfies the orthogonality relations along the linear mapping $\theta \times \theta: \Xi \times \Xi \rightarrow \mathfrak{m} \times \mathfrak{m}$ (see [BB10c, Lemma 3.8(1)]). \square

Remark 2.13. If $a_1, a_2 \in \mathcal{S}'(\Xi)$ and the operator product $T_{a_1} T_{a_2}$ is well defined in $L^2(\Xi)$, so that $\text{Op}^\theta(a_1) \text{Op}^\theta(a_2) = V^* T_{a_1} T_{a_2} V \in \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$ is well defined, then the Moyal product $a_1 \#^\theta a_2 \in \mathcal{S}'(\Xi^*)$ makes sense and we have

$$C_{a_1 \#^\theta a_2}(X, Z) = \int_{\Xi} C_{a_1}(X, Y) C_{a_2}(Y, Z) dY \quad (2.9)$$

for all $X, Z \in \Xi$. In fact, it follows by [BB10c, Lemma 3.19(4)] that for $X \in \Xi$ we have the integral $\text{Op}^\theta(a_2) \phi_X = \int_{\Xi} (\text{Op}^\theta(a_2) \phi_X \mid \phi_Y) \phi_Y dY$ convergent in $\mathcal{H}_{-\infty}$.

Therefore, if $\text{Op}^\theta(a_2) \phi_X \in \mathcal{D}(\text{Op}^\theta(a_1))$, then

$$\text{Op}^\theta(a_1) \text{Op}^\theta(a_2) \phi_X = \int_{\Xi} (\text{Op}^\theta(a_2) \phi_X \mid \phi_Y) \text{Op}^\theta(a_1) \phi_Y dY.$$

Hence we have

$$(\text{Op}^\theta(a_1) \text{Op}^\theta(a_2) \phi_X \mid \phi_Z) = \int_{\Xi} (\text{Op}^\theta(a_2) \phi_X \mid \phi_Y) (\text{Op}^\theta(a_1) \phi_Y \mid \phi_Z) dY$$

for all $X, Z \in \Xi$. \square

The next result is a generalization of the version of [HTW07, Prop. 0.1] without weights, which is recovered in the special case when π is the Schrödinger representation of the Heisenberg group.

Corollary 2.14. *Let $\phi \in \mathcal{H}_\infty$ and $p_1, p_2, p, q_1, q_2, q \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q}$. Then the Moyal product $\#^\theta$ defines a continuous bilinear map*

$$M_{\mathcal{W}(\phi, \phi)}^{p_1, q_1}(\pi^\#, \theta \times \theta) \times M_{\mathcal{W}(\phi, \phi)}^{p_2, q_2}(\pi^\#, \theta \times \theta) \rightarrow M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta).$$

Proof. For $j = 1, 2$ let $a_j \in M_{\mathcal{W}(\phi, \phi)}^{p_j, q_j}(\pi^\#, \theta \times \theta)$. We shall use the notation of Definition 2.11.

Then there exists $\beta_j \in L^{q_j}(\Xi)$ such that $\|C_{a_j}(\cdot, \cdot + Y)\|_{L^{p_j}(\Xi)} \leq \beta_j(Y)$ for every $Y \in \Xi$. On the other hand, it follows by (2.9) that for $X_1, X_2 \in \Xi$ we have

$$\begin{aligned} C_{a_1 \#^\theta a_2}(X_1, X_1 + X_2) &= \int_{\Xi} C_{a_1}(X_1, Y) C_{a_2}(Y, X_1 + X_2) dY \\ &= \int_{\Xi} C_{a_1}(X_1, X_1 + Y) C_{a_2}(X_1 + Y, X_1 + X_2) dY \end{aligned}$$

hence by Minkowski's inequality, and then Hölder's inequality, we get

$$\begin{aligned} \|C_{a_1 \#^\theta a_2}(\cdot, \cdot + X_2)\|_{L^p(\Xi)} &\leq \int_{\Xi} \|C_{a_1}(\cdot, \cdot + Y) C_{a_2}(\cdot + Y, \cdot + X_2)\|_{L^p(\Xi)} dY \\ &\leq \int_{\Xi} \|C_{a_1}(\cdot, \cdot + Y)\|_{L^{p_1}(\Xi)} \|C_{a_2}(\cdot + Y, \cdot + X_2)\|_{L^{p_2}(\Xi)} dY \\ &\leq \int_{\Xi} \|C_{a_1}(\cdot, \cdot + Y)\|_{L^{p_1}(\Xi)} \|C_{a_2}(\cdot, \cdot + X_2 - Y)\|_{L^{p_2}(\Xi)} dY \\ &\leq \int_{\Xi} \beta_1(Y) \beta_2(X_2 - Y) dY \\ &=: \beta(X_2) \end{aligned}$$

Since $\beta_j \in L^{q_j}(\Xi)$ for $j = 1, 2$, we have $\beta \in L^q(\Xi)$ and

$$\|a_1 \#^\theta a_2\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)} \leq \|\beta\|_{L^q(\Xi)} \leq \|\beta_1\|_{L^{q_1}(\Xi)} \|\beta_2\|_{L^{q_2}(\Xi)}.$$

By using Corollary 2.9, it then follows that $a_1 \#^\theta a_2 \in M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)$ and

$$\|a_1 \#^\theta a_2\|_{M_{\mathcal{W}(\phi, \phi)}^{p, q}(\pi^\#, \theta \times \theta)} \leq \|a_1\|_{M_{\mathcal{W}(\phi, \phi)}^{p_1, q_1}(\pi^\#, \theta \times \theta)} \|a_2\|_{M_{\mathcal{W}(\phi, \phi)}^{p_2, q_2}(\pi^\#, \theta \times \theta)},$$

which ends the proof. \square

The abstract version of Sjöstrand's algebra. We can now prove the main result of the paper.

Theorem 2.15. *If $\phi \in \mathcal{H}_\infty$, then the following assertions hold:*

- (1) *For every $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ we have $\text{Op}^\theta(a) \in \mathcal{B}(\mathcal{H})$ and moreover $\|\text{Op}^\theta(a)\| \leq \|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)}$.*
- (2) *The Moyal product $\#^\theta$ makes the modulation space $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ into an involutive associative Banach algebra.*
- (3) *Let*

$$\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta) = \mathbb{C}1 + M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta). \quad (2.10)$$

If $a_0 \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ and the operator $\text{Op}^\theta(a_0)$ is invertible in $\mathcal{B}(\mathcal{H})$, then there exists $b_0 \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ such that $\text{Op}^\theta(a_0)^{-1} = \text{Op}^\theta(b_0)$.

Proof. To prove Assertion (1), let $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ arbitrary and denote $\phi_X := \pi(\exp_M(\theta(X)))\phi \in \mathcal{H}_\infty$ for each $X \in \Xi$. It follows by Corollary 2.10 that there exists $\beta_a \in L^1(\Xi)$ such that $\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)} = \|\beta_a\|_{L^1(\Xi)}$ and

$$(\forall X_1, X_2 \in \Xi) \quad |(\text{Op}^\theta(a)\phi_{X_1} \mid \phi_{X_2})| \leq \beta_a(X_1 - X_2).$$

Now let $f \in \mathcal{H}_\infty$ and recall from [BB10c, Lemma 3.19] that $f = \int_{\Xi} (f | \phi_X) \phi_X dX$, hence

$$\begin{aligned} |(\text{Op}^\theta(a)f | \phi_Y)| &\leq \int_{\Xi} |(f | \phi_X)| \cdot |(\text{Op}^\theta(a)\phi_X | \phi_Y)| dX \\ &\leq \int_{\Xi} |(f | \phi_X)| \cdot \beta_a(X - Y) dX. \end{aligned}$$

That is, $|(\mathcal{A}_\phi^{\pi, \theta}(\text{Op}^\theta(a)f))(Y)| \leq (|\mathcal{A}_\phi^{\pi, \theta}f| * \beta_a(-\cdot))(Y)$ for all $Y \in \Xi$. Therefore,

$$\begin{aligned} \|\text{Op}^\theta(a)f\| &= \|(\mathcal{A}_\phi^{\pi, \theta}(\text{Op}^\theta(a)f))\|_{L^2(\Xi)} \leq \|\mathcal{A}_\phi^{\pi, \theta}f\|_{L^2(\Xi)} \|\beta_a\|_{L^1(\Xi)} \\ &= \|f\| \cdot \|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)}. \end{aligned}$$

Since $f \in \mathcal{H}_\infty$ is arbitrary and \mathcal{H}_∞ is dense in \mathcal{H} , the assertion follows.

For Assertion (2), to see that $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ is closed under the Moyal product, just use Corollary 2.14 for $p_1 = p_2 = p$ and $q_1 = q_2 = q$. Next note that if $\|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)} = 0$, then $\mathcal{A}_\phi^{\pi^\#, \theta \times \theta}a = 0$, and then it is straightforward to check that $a = 0$. To prove that the norm of $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ is complete, it suffices to check that any Cauchy sequence $\{a_j\}_{j \geq 1}$ has a convergent subsequence. By selecting a suitable subsequence, we may assume that $\|a_{j+1} - a_j\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)} < \frac{1}{2^j}$ for every $j \geq 0$, where $a_0 := 0$. It follows by Corollary 2.10 that there exists $\beta_{j+1} \in L^1(\Xi)$ such that $\|\beta_{j+1}\|_{L^1(\Xi)} < \frac{1}{2^j}$ and

$$(\forall X_1, X_2 \in \Xi) \quad |(\text{Op}^\theta(a_{j+1} - a_j)\phi_{X_1} | \phi_{X_2})| \leq \beta_{j+1}(X_1 - X_2). \quad (2.11)$$

Note that $\beta := \sum_{j=1}^{\infty} \beta_j \in L^1(\Xi)$ and, by summing up the above inequalities for $j = 0, \dots, k-1$ we get

$$(\forall X_1, X_2 \in \Xi) \quad |(\text{Op}^\theta(a_k)\phi_{X_1} | \phi_{X_2})| \leq (\beta_1 + \dots + \beta_k)(X_1 - X_2) \leq \beta(X_1 - X_2).$$

On the other hand, since $\{a_k\}_{k \geq 1}$ is a Cauchy sequence in $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$, it follows by Assertion (1) that there exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that $\lim_{k \rightarrow \infty} \|\text{Op}^\theta(a_k) - T\| = 0$. It follows by the above inequalities for $k \rightarrow \infty$ that

$$(\forall X_1, X_2 \in \Xi) \quad |(T\phi_{X_1} | \phi_{X_2})| \leq \beta(X_1 - X_2).$$

Moreover, it follows by Remark 2.4 that $T = \text{Op}^\theta(a)$ for some $a \in \mathcal{S}'(\Xi^*)$, and then $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ by the above inequality along with Corollary 2.10. Finally, by summing up the inequalities (2.11) for $j = k, k+1, \dots$ and using Corollary 2.10 again, we get $\|a - a_k\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)} \leq \sum_{j=k}^{\infty} \frac{1}{2^j} = \frac{1}{2^{k-1}}$ for arbitrary $k \geq 1$, hence

$$a = \lim_{k \rightarrow \infty} a_k \text{ in } M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta).$$

For Assertion (3) let $a_0 \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ and assume that the operator $\text{Op}^\theta(a_0)$ is invertible in $\mathcal{B}(\mathcal{H})$. There exist $\alpha \in \mathbb{C}$ and $a_{00} \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ such that $a_0 = \alpha + a_{00}$. We shall use the notation of Definition 2.11 and also recall that for the symbol $1 \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ we get the operator $T_1 \in \mathcal{B}(L^2(\Xi))$ with the properties $T_1 = T_1^* = (T_1)^2$ and $\text{Ran } T_1 = \text{Ran } V$. Moreover, for every

$a \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ we have $T_a T_1 = T_1 T_a = T_a$, and in particular T_a vanishes on $(\text{Ran } T_1)^\perp$.

It then follows that if $z \in \mathbb{C} \setminus \{\alpha\}$ and the operator

$$z\mathbf{1} - \text{Op}^\theta(a_0) = \text{Op}^\theta(z - a_0) = \text{Op}^\theta(z - \alpha - a_{00})$$

is invertible in $\mathcal{B}(\mathcal{H})$, then $(z - \alpha)(\mathbf{1} - T_1) + T_{z - \alpha - a_{00}} = (z - \alpha)\mathbf{1} - T_{a_{00}}$ is invertible in $\mathcal{B}(L^2(\Xi))$. On the other hand, since $a_{00} \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$, it follows by Remark 2.10 that there exists $\beta_0 \in L^1(\Xi)$ such that the integral kernel $C_{a_{00}}$ of $T_{a_{00}}$ satisfies the estimate $|C_{a_{00}}(X - Y)| \leq \beta_0(X - Y)$ for all $X, Y \in \Xi$. We then get by [Ku99, Th. 5.4.7] (see also [Ku01]) that $((z - \alpha)\mathbf{1} - T_{a_{00}})^{-1} = (z - \alpha)^{-1}\mathbf{1} - N_z$, where $N_z \in \mathcal{B}(L^2(\Xi))$ is an integral operator whose kernel K_{N_z} satisfies a similar estimate $|K_{N_z}(X - Y)| \leq \beta_z(X - Y)$ for all $X, Y \in \Xi$ and a suitable function $\beta_z \in L^1(\Xi)$. Since $T_{a_{00}} T_1 = T_1 T_{a_{00}} = T_{a_{00}}$, it follows that $N_z T_1 = T_1 N_z = N_z$. By using the fact that $\text{Op}^\theta: \mathcal{S}'(\Xi^*) \rightarrow \mathcal{L}(\mathcal{H}_\infty, \mathcal{H}_{-\infty})$ is a linear isomorphism (see [BB10c, Rem. 3.11]) and Lemma 2.12, we then get $b_z \in \mathcal{S}'(\Xi^*)$ such that $\text{Op}^\theta(b_z) \in \mathcal{B}(\mathcal{H})$ and $T_{b_z} = N_z$. Moreover, the estimates satisfied by the integral kernel of N_z show that actually $b_z \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ by Remark 2.10 again.

We have thus shown that if $z \in \mathbb{C} \setminus \{\alpha\}$ and $z\mathbf{1} - \text{Op}^\theta(a_0)$ (which is equal to $\text{Op}^\theta(z - a_0)$) is invertible in $\mathcal{B}(\mathcal{H})$, then there exists $b_z \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ such that $\text{Op}^\theta(z - a_0)^{-1} = \text{Op}^\theta((z - \alpha)^{-1} - b_z)$. Thus we can see that $z - a_0$ is invertible in the unital Banach algebra $\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ and its inverse is $(z - \alpha)^{-1} - b_z$. In particular, $z \mapsto b_z$ is a holomorphic mapping from the complement of the spectrum of $\text{Op}^\theta(a_0)$ into $\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$. Now, since $\text{Op}^\theta(a_0) \in \mathcal{B}(\mathcal{H})$ is an invertible operator, there exists a piecewise smooth closed curve that does not contain α and surrounds the spectrum of $\text{Op}^\theta(a_0)$, and we have by holomorphic functional calculus

$$\text{Op}^\theta(a_0)^{-1} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} (z - \text{Op}^\theta(a_0))^{-1} dz.$$

Since $\mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta)$ is a unital Banach algebra, we can define

$$b_0 := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} ((z - \alpha)^{-1} - b_z) dz \in \mathcal{M}_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^\#, \theta \times \theta).$$

Then

$$\begin{aligned} \text{Op}^\theta(b_0) &= \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} \text{Op}^\theta((z - \alpha)^{-1} - b_z) dz = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} (z - \text{Op}^\theta(a_0))^{-1} dz \\ &= \text{Op}^\theta(a_0)^{-1}, \end{aligned}$$

which completes the proof. \square

Remark 2.16. A more general result on the continuity of the operators $\text{Op}^\theta(a)$ on modulation spaces was obtained in [BB10c] by a completely different method based on continuity properties of the cross-Wigner distribution. \square

3. APPLICATIONS TO THE MAGNETIC WEYL CALCULUS

Notation 3.1. Let G be a simply connected, nilpotent Lie group with the Lie algebra \mathfrak{g} and the inverse of the exponential map denoted by $\log_G: G \rightarrow \mathfrak{g}$. We denote by $\lambda: G \rightarrow \text{End}(\mathcal{C}^\infty(G))$, $g \mapsto \lambda_g$, the left regular representation defined by $(\lambda_g \phi)(x) = \phi(g^{-1}x)$ for every $x, g \in G$ and $\phi \in \mathcal{C}^\infty(G)$. Moreover, we denote by $\mathbf{1}$ the constant function which is identically equal to 1 on G . (This should not be confused with the unit element of G , which is denoted in the same way.)

If the space of globally defined smooth vector fields on G (that is, global sections in its tangent bundle) is denoted by $\mathfrak{X}(G)$ and the space of globally defined smooth 1-forms (that is, global sections in its cotangent bundle) is denoted by $\Omega^1(G)$, then there exists a natural bilinear map

$$\langle \cdot, \cdot \rangle: \Omega^1(G) \times \mathfrak{X}(G) \rightarrow \mathcal{C}^\infty(G)$$

defined as usually by evaluations at every point of G .

For arbitrary $g \in G$ we denote the corresponding right-translation mapping by $R_g: G \rightarrow G$, $h \mapsto hg$. Then we define the injective linear mapping

$$\iota^R: \mathfrak{g} \rightarrow \mathfrak{X}(G)$$

by $(\iota^R X)(g) = (T_1(R_g))X \in T_g G$ for all $g \in G$ and $X \in \mathfrak{g}$.

Moreover, we define

$$\Xi = \Xi^* := \mathfrak{g} \times \mathfrak{g}^*$$

and the symplectic duality pairing

$$\langle \cdot, \cdot \rangle: \Xi^* \times \Xi \rightarrow \mathbb{R}, \quad ((X_1, \xi_1), (X_2, \xi_2)) \mapsto \langle \xi_1, X_2 \rangle - \langle \xi_2, X_1 \rangle$$

where $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ is the natural duality pairing. \square

Setting 3.2. Throughout this section we denote by \mathcal{F} a linear space of real functions on the Lie group G which is endowed with a sequentially complete, locally convex topology and satisfies the following conditions:

- (1) The linear space \mathcal{F} is invariant under the representation of G by left translations, that is, if $\phi \in \mathcal{F}$ and $g \in G$ then $\lambda_g \phi \in \mathcal{F}$.
- (2) There exist the continuous inclusion maps $\mathfrak{g}^* \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{C}_{\text{pol}}^\infty(G)$, where the embedding $\mathfrak{g}^* \hookrightarrow \mathcal{F}$ is given by $\xi \mapsto \xi \circ \log_G$.
- (3) The mapping $G \times \mathcal{F} \rightarrow \mathcal{F}$, $(g, \phi) \mapsto \lambda_g \phi$ is smooth. For every $\phi \in \mathcal{F}$ we denote by $\dot{\lambda}(\cdot)\phi: \mathfrak{g} \rightarrow \mathcal{F}$ the differential of the mapping $g \mapsto \lambda_g \phi$ at the point $\mathbf{1} \in G$.

For instance, the function space $\mathcal{C}_{\text{pol}}^\infty(G)$ is admissible. Here $\mathcal{C}_{\text{pol}}^\infty(G)$ is the space of smooth functions $\phi: G \rightarrow \mathbb{R}$ such that the function $\phi \circ \log_G: \mathfrak{g} \rightarrow \mathbb{R}$ and its partial derivatives have polynomial growth. \square

Definition 3.3. We define the semidirect product $M = \mathcal{F} \rtimes_\lambda G$, which is a locally convex Lie group, and the unitary representation

$$\pi: M \rightarrow \mathcal{B}(L^2(G)), \quad \pi(\phi, g)f = e^{i\phi} \lambda_g f \text{ for } \phi \in \mathcal{F}, g \in G, \text{ and } f \in L^2(G).$$

If we have $A \in \Omega^1(G)$ with \mathcal{F} -growth, in the sense that $\langle A, \iota^R X \rangle \in \mathcal{F}$ whenever $X \in \mathfrak{g}$, then we define the linear mapping

$$\theta^A: \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathfrak{m} = \mathcal{F} \rtimes_\lambda \mathfrak{g}, \quad (X, \xi) \mapsto (\xi \circ \log_G + \langle A, \iota^R X \rangle, X).$$

\square

Remark 3.4. The representation π is twice nuclearly smooth and its space of smooth vectors is the Schwartz space $\mathcal{S}(G)$ ([BB10c, Cor.4.5]) and the following assertions hold for every 1-form $A \in \Omega^1(G)$ with \mathcal{F} -growth:

- (1) The representation π satisfies the orthogonality relations along the mapping θ^A .
- (2) The representation π satisfies the density condition along θ^A .
- (3) The localized Weyl calculus for π along θ^A is regular and defines a unitary operator $\text{Op}^{\theta^A} : L^2(\mathfrak{g} \times \mathfrak{g}^*) \rightarrow \mathfrak{S}_2(L^2(G))$.
- (4) If $u \in \text{U}(\mathfrak{m}_{\mathbb{C}})$ and $\phi \in \mathcal{S}(G)$, the function $\|\text{d}\pi(\text{Ad}_{\text{U}(\mathfrak{m}_{\mathbb{C}})}(\exp_M(\theta^A(\cdot)))u)\phi\|$ has polynomial growth on $\mathfrak{g} \times \mathfrak{g}^*$.

These properties have been established in [BB10c, Cor.4.5], and it thus follows that all of the conditions of Setting 2.1 are satisfied in the present setting provided by the representation π and the linear mapping θ^A .

Just as in [BB09a] we shall denote the corresponding Moyal product $\#^{\theta^A}$ simply by θ^A and localized Weyl calculus for π along θ^A is denoted by $\text{Op}^A(\cdot)$ and is called the *magnetic Weyl calculus* associated with the magnetic potential $A \in \Omega^1(G)$. The corresponding *magnetic field* is $B := \text{d}A \in \Omega^2(G)$. \square

Theorem 3.5. *If $\phi \in \mathcal{S}(G)$, then the following assertions hold:*

- (1) *For every $a \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^{\#}, \theta^A \times \theta^A)$ we have $\text{Op}^A(a) \in \mathcal{B}(L^2(G))$ and moreover $\|\text{Op}^A(a)\| \leq \|a\|_{M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^{\#}, \theta^A \times \theta^A)}$.*
- (2) *The Moyal product $\#^A$ makes the modulation space $M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^{\#}, \theta^A \times \theta^A)$ into an associative Banach algebra.*
- (3) *If $a_0 \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^{\#}, \theta^A \times \theta^A)$ and $\text{Op}^A(a_0) \in \mathcal{B}(L^2(G))$ is invertible, then there exists $b_0 \in M_{\mathcal{W}(\phi, \phi)}^{\infty, 1}(\pi^{\#}, \theta^A \times \theta^A)$ such that $\text{Op}^A(a_0)^{-1} = \text{Op}^A(b_0)$.*

Proof. The above Remark 3.4 shows that Corollary 2.14 and Theorem 2.15 apply, and then the assertions follow. \square

4. APPLICATIONS TO REPRESENTATIONS OF NILPOTENT LIE GROUPS

All of the conditions of Setting 2.1 are satisfied if M is a finite-dimensional nilpotent Lie group, π is a unitary irreducible representation with the corresponding coadjoint orbit Ξ^* , Ξ is a predual of the coadjoint orbit Ξ (in the sense of [BB10b]) and $\theta: \Xi \hookrightarrow \mathfrak{m}$ is the embedding map. This will be the setting of the present section, and our point here is to describe how the abstract results of Section 2 can be specialized in this framework, and also to point out how they can be further sharpened in the special case when π is a square-integrable representation modulo the center.

Setting 4.1. Throughout this section we shall use the following notation:

- (1) Let G be a connected, simply connected, nilpotent Lie group with Lie algebra \mathfrak{g} . Then the exponential map $\exp_G: \mathfrak{g} \rightarrow G$ is a diffeomorphism with the inverse denoted by $\log_G: G \rightarrow \mathfrak{g}$.
- (2) We denote by \mathfrak{g}^* the linear dual space to \mathfrak{g} and by $\langle \cdot, \cdot \rangle: \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ the natural duality pairing.
- (3) Let $\xi_0 \in \mathfrak{g}^*$ with the corresponding coadjoint orbit $\mathcal{O} := \text{Ad}_G^*(G)\xi_0 \subseteq \mathfrak{g}^*$.
- (4) Let $\pi: G \rightarrow \mathcal{B}(\mathcal{H})$ be any unitary irreducible representations associated with the coadjoint orbit \mathcal{O} by Kirillov's theorem ([Ki62]).

- (5) The *isotropy group* at ξ_0 is $G_{\xi_0} := \{g \in G \mid \text{Ad}_G^*(g)\xi_0 = \xi_0\}$ with the corresponding *isotropy Lie algebra* $\mathfrak{g}_{\xi_0} = \{X \in \mathfrak{g} \mid \xi_0 \circ \text{ad}_{\mathfrak{g}} X = 0\}$. If we denote the *center* of \mathfrak{g} by $\mathfrak{z} := \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] = \{0\}\}$, then $\mathfrak{z} \subseteq \mathfrak{g}_{\xi_0}$.
- (6) Let $n := \dim \mathfrak{g}$ and fix a sequence of ideals in \mathfrak{g} ,

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$$

such that $\dim(\mathfrak{g}_j/\mathfrak{g}_{j-1}) = 1$ and $[\mathfrak{g}, \mathfrak{g}_j] \subseteq \mathfrak{g}_{j-1}$ for $j = 1, \dots, n$.

- (7) Pick any $X_j \in \mathfrak{g}_j \setminus \mathfrak{g}_{j-1}$ for $j = 1, \dots, n$, so that the set $\{X_1, \dots, X_n\}$ will be a *Jordan-Hölder basis* in \mathfrak{g} .

Also consider the set of *jump indices* of the coadjoint orbit \mathcal{O} with respect to the aforementioned Jordan-Hölder basis,

$$e := \{j \in \{1, \dots, n\} \mid \mathfrak{g}_j \not\subseteq \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0}\} = \{j \in \{1, \dots, n\} \mid X_j \notin \mathfrak{g}_{j-1} + \mathfrak{g}_{\xi_0}\}$$

and then define the corresponding *predual of the coadjoint orbit* \mathcal{O} ,

$$\mathfrak{g}_e := \text{span}\{X_j \mid j \in e\} \subseteq \mathfrak{g}.$$

We note the direct sum decomposition $\mathfrak{g} = \mathfrak{g}_{\xi_0} \dot{+} \mathfrak{g}_e$. \square

Theorem 4.2. *If the representation π is square integrable modulo the center, then the following assertions hold:*

- (1) *If $p_1, p_2, p, q_1, q_2, q \in [1, \infty]$ satisfy the conditions $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ and $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q}$, then the Moyal product $\#^\theta$ defines a continuous bilinear map*

$$M^{p_1, q_1}(\pi^\#) \times M^{p_2, q_2}(\pi^\#) \rightarrow M^{p, q}(\pi^\#).$$

- (2) *The Moyal product $\#^\theta$ makes the modulation space $M^{\infty, 1}(\pi^\#)$ into an associative involutive Banach algebra and the Weyl calculus defines an injective continuous $*$ -homomorphism $\text{Op}: M^{\infty, 1}(\pi^\#) \rightarrow \mathcal{B}(\mathcal{H})$.*

- (3) *If $a_0 \in M^{\infty, 1}(\pi^\#)$ and $\text{Op}^\theta(a_0) \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then there exists $b_0 \in M^{\infty, 1}(\pi^\#)$ such that $\text{Op}^\theta(a_0)^{-1} = \text{Op}^\theta(b_0)$.*

Proof. Recall that the modulation spaces of symbols $M^{p, q}(\pi^\#)$ are independent on the choice of a window vector by [BB10b, Example 3.4(2)]. Then the assertions follow by the above Corollary 2.14 and Theorem 2.15. \square

Theorem 4.3. *Assume that the representation π is square integrable modulo the center of G and let $\mathfrak{g}_e = \mathfrak{g}_e^1 \dot{+} \mathfrak{g}_e^2$ be any decomposition of the predual into a direct sum of linear subspaces. If $\phi \in \mathcal{H}_\infty$ and we have $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, then $M_\phi^{p_1, q_1}(\pi) \subseteq M_\phi^{p_2, q_2}(\pi)$.*

Proof. It follows by (2.1) that the proof will be complete as soon as we have proved that if $p, q \in [1, \infty]$ and $f \in M_\phi^{p, q}(\pi)$, then $f \in M_\phi^{\infty, \infty}(\pi)$, that is, $\mathcal{A}_\phi f \in L^\infty(\mathfrak{g}_e)$.

In fact, let us define

$$R_\phi: \mathfrak{g}_e \times \mathfrak{g}_e \rightarrow \mathbb{C}, \quad R_\phi(X, Y) = (\pi(\exp_G X)\phi \mid \pi(\exp_G Y)\phi) = (\mathcal{A}_\phi(\pi(\exp_G \phi)))(Y).$$

Now let us denote by $*_e$ the Baker-Campbell-Hausdorff multiplication on the nilpotent Lie algebra $\mathfrak{g}_e \simeq \mathfrak{g}/\mathfrak{z}$. There exists a polynomial map $\alpha: \mathfrak{g}_e \times \mathfrak{g}_e \rightarrow \mathbb{R}$ such that $\pi(\exp_G((-X) * Y)) = e^{i\alpha(-X, Y)}\pi(\exp_G((-X) *_e Y))$ (see for instance [Ma07]), hence

$$\begin{aligned} R_\phi(X, Y) &= e^{-i\alpha(-X, Y)}(\phi \mid \pi(\exp_G((-X) *_e Y))\phi) \\ &= e^{-i\alpha(-X, Y)}(\mathcal{A}_\phi \phi)((-X) *_e Y). \end{aligned}$$

Since $\mathcal{A}_\phi \phi \in \mathcal{S}(\mathfrak{g}_e)$ (see [Pe94]) and the Lebesgue measure on \mathfrak{g}_e coincides with the Haar measure on the nilpotent Lie group $(\mathfrak{g}_e, *_e)$, it then follows that

$$(\forall r, s \in [1, \infty]) \quad \sup_{X \in \mathfrak{g}_e} \|R(X, \cdot)\|_{L^{r,s}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)} < \infty. \quad (4.1)$$

On the other hand, note that $R_\phi(X, Y) = (\mathcal{A}_\phi(\pi(\exp_G \phi)))(Y)$, hence

$$\begin{aligned} (\mathcal{A}_\phi f)(X) &= (f \mid \pi(\exp_G X) \phi) \\ &= (f \mid \int_{\mathfrak{g}_e} (\mathcal{A}_\phi(\pi(\exp_G \phi)))(Y) \pi(\exp_G Y) \phi dY) \\ &= (f \mid \int_{\mathfrak{g}_e} R_\phi(X, Y) \pi(\exp_G Y) \phi dY) \\ &= \int_{\mathfrak{g}_e} \overline{R_\phi(X, Y)} (f \mid \pi(\exp_G Y) \phi) dY \end{aligned}$$

whence

$$(\forall X \in \mathfrak{g}_e) \quad (\mathcal{A}_\phi f)(X) = (\mathcal{A}_\phi f \mid R(X, \cdot)), \quad (4.2)$$

where the right-hand side makes sense since $\mathcal{A}_\phi f \in \mathcal{C}(\mathfrak{g}_e) \cap \mathcal{S}'(\mathfrak{g}_e)$ by [BB10b, Cor. 2.9(1)], while $R(X, \cdot) \in \mathcal{S}(\mathfrak{g}_e)$ by [Pe94]. If $f \in M_\phi^{p,q}(\pi)$, then it follows by (4.2) along with Hölder's inequality in mixed-norm spaces (see [BP61]) and (4.1) that

$$\sup_{X \in \mathfrak{g}_e} |(\mathcal{A}_\phi f)(X)| \leq \sup_{X \in \mathfrak{g}_e} (\|\mathcal{A}_\phi f\|_{L^{p,q}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)} \|R(X, \cdot)\|_{L^{p',q'}(\mathfrak{g}_e^1 \times \mathfrak{g}_e^2)}) < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Thus $\mathcal{A}_\phi f \in L^\infty(\mathfrak{g}_e)$, and this completes the proof, in view of the beginning remark. \square

Corollary 4.4. *If the representation π is square integrable modulo the center of G , then for every $p \in [1, \infty]$ the modulation space $M^{p,1}(\pi^\#)$ is a subalgebra of $M^{\infty,1}(\pi^\#)$ endowed with the Moyal product $\#$.*

Proof. As noted in [BB10b, Example 3.4(2) and Remark 3.7], the representation $\pi^\# : G \ltimes G \rightarrow \mathcal{B}(L^2(\mathcal{O}))$ is square integrable (modulo the center) and its modulation spaces are independent on the choice of the window vector. Thus the above Theorem 4.3 applies for the representation $\pi^\#$ instead of π , and it follows that $M^{p_1,q_1}(\pi^\#) \subseteq M^{p_2,q_2}(\pi^\#)$ whenever $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$.

In particular we have $M^{p,1}(\pi^\#) \subseteq M^{\infty,1}(\pi^\#)$ if $1 \leq p \leq \infty$. Moreover, it follows by Corollary 2.14 that if $a_1, a_2 \in M^{p,1}(\pi^\#)$, then $a_1 \# a_2 \in M^{\frac{p}{2},1}(\pi^\#) \subseteq M^{p,1}(\pi^\#)$, and this completes the proof. \square

In the special case when π is the Schrödinger representation of the Heisenberg group, the above result goes back to [To01]; see also [HTW07]. We also note that in this case we have $\mathcal{M}^{\infty,1}(\pi^\#) = M^{\infty,1}(\pi^\#)$.

Acknowledgment. The second-named author acknowledges partial financial support from the Project MTM2007-61446, DGI-FEDER, of the MCYT, Spain, and from the grant PNII - Programme ‘‘Idei’’ (code 1194).

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